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Multilayered Shell Theories Accounting for Layerwise Mixed Description, Part 1: Governing Equations

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A Reissner mixed variational equation is employed in this paper to derive the differential governing equations of multilayered, double curved shells made of orthotropic laminae in linear static cases. A layerwise description is referred to by assuming two independent fields in the thickness direction for the transverse stress (both shear and normal components) and displacement variables in each layer. Interlaminar values are used as the unknown variables of the introduced expansions. The continuity conditions of displacements and transverse shear and normals stresses at the interfaces between two consecutive layers, referred to as C_z^0 requirements, have been a priori fulfilled. These have been used to drive the governing equations from a layer to a multilayered level. Classical displacement formulations and related equivalent single-layer equations have been derived for comparison purposes. No assumptions have been made concerning the terms of type thickness to radii shell ratio h/R. Donnell's shallow shell-type equations are given as particular cases for all of the considered theories. Indicial notations and arrays have been used extensively to handle the presented developments in a concise manner. Numerical evaluations and comparisons to exact and other available two-dimensional solutions are given in a companion paper (E. Carrera, "Multilayered Shell Theories Accounting for Layerwise Mixed Description, Part 2: Numerical Evaluations," *AIAA Journal*, Vol. 37, No. 9, 1999, pp. 1117–1124).

I. Introduction

IN recent years, considerable attention has been paid to the development of appropriate two-dimensional shell theories that can accurately describe the response of multilayered anisotropic thick shells. In fact, thick shell component analyses and fatigue design require an accurate description of local stress fields to include highly accurate assessment of localized regions where damage is likely to take place. Examples of multilayered shell structures used in modern aerospace vehicles are laminated constructions made of anisotropic composite materials, sandwich panels, layered structures used as thermal protection, or intelligent structural system embedding piezolayers.

It was pointed out by Koiter[1] that, for traditional isotropic one-layer shells, refinements of Love's first approximation theory are meaningless unless the effects of transverse shear and normal stress are both taken into account in a refined theory. Layered shells deserve special attention. These are characterized by a noncontinuous thermomechanical material property distribution in the thickness direction and further requisites become essential for a reliable modeling of such structures. Among these, the fulfillment of both continuity of displacement and transverse shear and normal stresses at the interface between two adjacent layers is such a necessary desideratum. In Ref. 2 these requisites are referred to as C_z^0 requirements that state that both displacements and transverse stress components are C^0-continuous functions in the thickness shell coordinate z. An increasing role is played by the C_z^0 requirements and by Koiter's recommendation in the case of laminated shells made of composite materials presently used in aerospace structures. These materials exhibit higher values of Young's moduli orthotropic ratio ($E_L/E_T = E_L/E_z = 5 \div 40$, L denotes the fiber directions, whereas T and z are two-direction orthogonal to L) and the lower transverse shear moduli ratio ($G_{LT}/E_L \approx G_{TT}/E_L = \frac{1}{10} \div \frac{1}{200}$) leading to higher transverse shear and normal stress deformability in comparison to isotropic cases. Approximated three-dimensional solutions by Noor and Rarig[3] and Noor and Peters[4,5] and more recent exact three-dimensional solutions by Ren[6] and Varadan and Bhaskar[7] have numerically confirmed the need of the previously mentioned refinements for static and dynamic cylindrical shell problems. In particular, the fundamental role played by transverse normal stress σ_{zz} was underlined. Nevertheless, three-dimensional elasticity solutions are only available in a very few cases and these are mainly related to simple geometries, a specific stacking sequence of the lamina, and linear problems. In the most general cases and to minimize the computational effort, two-dimensional models are preferred in practice.

Starting from the early work by Shtayerman,[8] many two-dimensional models have been proposed for anisotropic layered shells. The so-called axiomatic approach[9] (where a certain displacement or stress field is postulated in the shell thickness direction) and asymptotic methods[10–16] (where the three-dimensional equations are expanded in terms of an introduced shell parameter) have both been applied to derive simplified analysis. Exhaustive overviews on these topics can be found in many published review papers. Classical theories were reviewed by Bert.[17] An interesting overview, including works that appeared in Russian literature, can be found in the book by Librescu.[18] Recent developments in the Russian school concerning the fulfillment of the C_z^0 requirements were overviewed by Grigolyuk and Kulikov.[19] Reviews on finite element shell formulations can be found in the work by Dennis and Palazotto,[20] Merk,[21] and Di and Ramm.[22] Recent articles on the application of asymptotic methods to anisotropic shells can be found in Fettahlioglu and Steele,[23] Widera and Logan,[24] Widera and Fan,[25] and Spencer et al.[26] Two exhaustive and more recent surveys have been provided by Kapania[27] and Noor and Burton[28] that address a complete overview of different aspects of multilayered shells modelings. Herein, attention is focused on the axiomatic approach. A short review of this approach follows.

Classical displacement formulations start by assuming a linear or higher-order expansion for the displacement fields in the thickness direction. In-plane and transverse stresses are then computed by means of Hooke's law. According to this procedure, it is found that transverse stresses (both shear and normal components) are discontinuous at the interfaces. To overcome these difficulties, these stresses are evaluated a posteriori in most applications by implementing a postprocessing procedure, e.g., through the thickness integration of the three-dimensional indefinite equation of equilibrium. A few examples in which a layerwise model (LWM) description is used (the number of the unknowns depends on the number of layers) are works by Hsu and Wang,[29] Cheung and Wu,[30] and Barbero et al.[31] Others, in which an equivalent single-layer model

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(ESLM) description was preferred, are the works by Hildebrand et al.,[32] Whitney and Sun,[33] Reddy and Liu,[34] Librescu et al.,[35] and Dennis and Palazotto.[20,36] The interesting theory by Rath and Das[37] should be mentioned from the ESLM analyses, where interlaminar transverse shear continuity was a priori fulfilled in both the symmetrical and unsymmetrical case through the thickness response of layered shells. A particular example of the theory in Ref. 37 was analyzed in Ref. 38 for symmetrically laminated cylinders. The numerical analyses reported in the cited works conclude the following:

1) An a priori description of transverse stress cannot account for LWMs based on the displacement formulation.

2) Layerwise (LW) analyses usually lead to a better description than ESLM ones; such a superiority is more evident for arbitrarily laminated shells with increasing layers.

3) The ESLM analysis experienced difficulties in accurately describing a σ_{zz} and the related consequences.

4) The mentioned postprocessing procedure for the calculation of transverse stresses cannot be implemented for most of the available models in the general case of asymmetric in-plane displacement fields (i.e., two different results could be obtained for the stress distributions by starting from the top or from the bottom shell surface).

Reissner[39,40] proposed a mixed variational equation for the purpose of overcoming the impossibility of fulfilling a priori the interlaminar continuity for both transverse shear and normal stresses, which furnishes equilibrium and constitutive equations that are consistent with an assumed displacement and transverse stress field. Similar discussion and conclusions can be read in the overview paper by Grigolyuk and Kulikov.[19] This tool was applied to shells by Bhaskar and Varadan[41] and Jing and Tzeng[42] for the case of ESLM analysis. Both works neglected the transverse normal stress. Related results confirmed the already known conclusion for the plate cases: The use of Reissner's mixed variational equations associated to an ESLM description is not sufficient to describe accurately the σ_{zz} effects and related consequences. In particular, in Refs. 41 and 43 it was shown that stress computed by the assumed transverse stress field could very inaccurate. Therefore, the use of Reissner equation for an accurate evaluation of transverse stresses requires a layerwise description.

The convenience of referring to a Reissner mixed variational equation was shown in a recent series of articles related to multilayered plate analyses.[2,44–48] Excellent agreement with the three-dimensional exact solutions was found for both static and dynamic analysis. In particular, it was shown that the proposed layer-mixed description gives an excellent a priori description of the transverse shear and normal stress fields. These results encouraged the author to extend his research to the multilayered shell, which is presented in this paper.

It is a well-established result obtained from traditional isotropic shell structures analysis[49–52] that accurate two-dimensional shell modeling cannot come without an equivalently accurate description of the curvature terms. The neglectfulness of terms of type h/R (thickness to radii shell ratio) or the use of Donnell's shallow-shell type approximations could be very restrictive in thick-shell analysis. In fact, as shown by Soldatos,[53] Carrera,[54] and Jing and Tzeng,[42] any refinement related to the fulfillment of C_z^0 requirements would be meaningless unless curvature terms are well described. For this reason, no assumption will be introduced in this paper concerning curvature terms. For comparison purposes, Donnell's shallow shell equations are depicted in all of the considered cases with the use of a trace operator.

The paper has been organized as follows. Preliminary descriptions of the geometry and materials of the multilayered shells are given in Sec. II. Section III shows the displacement and stress fields introduced in each layer. The necessary variational statements are given in Sec. IV. Sections V and VI derive the governing equations for both mixed and displacement formulated theories. A technique to write multilayered equations is given in Sec. VII. The conclusions are drawn in Sec. VIII. Appendix A derives ESLM equations related to displacement formulations whereas Appendix B shows, in a simplified case, how the introduced indicial notation works. The numerical analysis is then given in Ref. 55. Extension to dynamics and results on free vibration response of multilayered shells have been provided in Ref. 56.

II. Preliminary Description

A. Multilayered Shell Geometry

The salient features of shell geometry are shown in Fig. 1. A laminated shell composed of N_l layers is considered. The integer k, used as a superscript or subscript, denotes the layer number that starts from the shell bottom. The layer geometry is denoted by the same symbols as those used for the whole multilayered shell and vice versa. Here a k-layer case whose geometries are described next is referred to. The α_k and β_k are the curvilinear orthogonal coordinates (coinciding with the lines of principal curvature) on the layer reference surface Ω_k (middle surface of the k layer). The z_k denotes the rectilinear coordinate in the direction normal to Ω_k. The Γ_k is the Ω_k boundary: Γ_k^g and Γ_k^m are those parts of Γ_k on which the geometrical and mechanical boundary conditions are imposed, respectively; these boundaries are herein considered parallel to α_k or β_h. For convenience the further dimensionless thickness coordinates are introduced $\zeta_k = 2z_k/h_k$, where h_k denotes the thickness in A_k domain.

The following relations hold in the given orthogonal system of curvilinear coordinates[49,51]:

square of line elements

$$\mathrm{d}s_k^2 = H_\alpha^k\,\mathrm{d}\alpha_k^2 + H_\beta^k\,\mathrm{d}\beta_k^2 + H_z^k\,\mathrm{d}z_k^2 \tag{1}$$

area of an infinitesimal rectangle on Ω_k

$$\mathrm{d}\Omega_k = H_\alpha^k H_\beta^k\,\mathrm{d}\alpha_k\,\mathrm{d}\beta_k \tag{2}$$

infinitesimal volume

$$\mathrm{d}V = H_\alpha^k H_\beta^k H_z^k\,\mathrm{d}\alpha_k\,\mathrm{d}\beta_k\,\mathrm{d}z_k \tag{3}$$

where

$$H_\alpha^k = A\left(1 + z_k/R_\alpha^k\right), \qquad H_\beta^k = B\left(1 + z_k/R_\beta^k\right), \qquad H_z^k = 1 \tag{4}$$

The R_α^k and R_β^k are the radii of curvature in the directions of α_k and β_k, respectively. The coefficients of the first fundamental form of Ω_k are A and B. For the sake of simplicity, attention is herein restricted to a shell with a constant curvature, i.e., a doubly curved shell (cylindrical, spherical, toroidal geometries) for which $A = B = 1$.

B. Classical and Mixed Form of Hooke's Law for Orthotropic Lamina

The lamina are considered to be homogeneous and to operate in the linear elastic range. By employing stiffness coefficients, Hooke's law for the anisotropic k lamina is written in the form

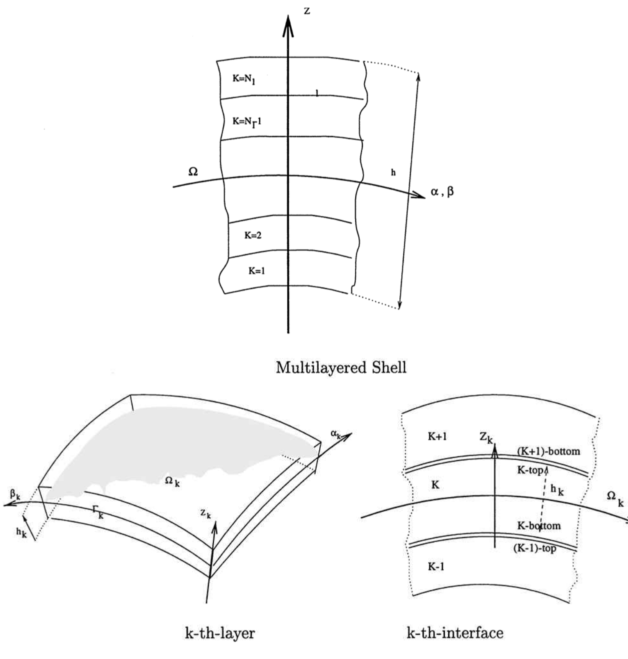

Fig. 1 Geometry and notation of multilayered shells.

$\sigma_i = \tilde{C}_{ij}\epsilon_j$, where the subindices i and j, ranging from 1 to 6, stand for the index couples 11, 22, 33, 13, 23, and 12, respectively. The material is assumed to be orthotropic, as specified, by $\tilde{C}_{14} = \tilde{C}_{24} = \tilde{C}_{34} = \tilde{C}_{64} = \tilde{C}_{15} = \tilde{C}_{25} = \tilde{C}_{35} = \tilde{C}_{65} = 0$ (Ref. 56). This implies that $\sigma^k_{\alpha z}$ and $\sigma^k_{\beta z}$ depend only on $\epsilon^k_{\alpha z}$ and $\epsilon^k_{\beta z}$. In matrix form,

$$\boldsymbol{\sigma}^k_{pH_d} = \tilde{\boldsymbol{C}}^k_{pp}\boldsymbol{\epsilon}^k_{pG} + \tilde{\boldsymbol{C}}^k_{pn}\boldsymbol{\epsilon}^k_{nG}, \qquad \boldsymbol{\sigma}^k_{nH_d} = \tilde{\boldsymbol{C}}^k_{np}\boldsymbol{\epsilon}^k_{pG} + \tilde{\boldsymbol{C}}^k_{nn}\boldsymbol{\epsilon}^k_{nG} \tag{5}$$

where

$$\tilde{\boldsymbol{C}}^k_{pp} = \begin{bmatrix} \tilde{C}^k_{11} & \tilde{C}^k_{12} & \tilde{C}^k_{16} \\ \tilde{C}^k_{12} & \tilde{C}^k_{22} & \tilde{C}^k_{26} \\ \tilde{C}^k_{16} & \tilde{C}^k_{26} & \tilde{C}^k_{66} \end{bmatrix}, \qquad \tilde{\boldsymbol{C}}^k_{pn} = \tilde{\boldsymbol{C}}^{k^T}_{np} = \begin{bmatrix} 0 & 0 & \tilde{C}^k_{13} \\ 0 & 0 & \tilde{C}^k_{23} \\ 0 & 0 & \tilde{C}^k_{36} \end{bmatrix}$$

$$\tilde{\boldsymbol{C}}^k_{nn} = \begin{bmatrix} \tilde{C}^k_{44} & \tilde{C}^k_{45} & 0 \\ \tilde{C}^k_{45} & \tilde{C}^k_{55} & 0 \\ 0 & 0 & \tilde{C}^k_{66} \end{bmatrix}$$

Boldfaced letters denote arrays. The superscript T signifies array transposition, and the subscripts n and p denote transverse (out-of-plane, normal) and in-plane values, respectively. Therefore

$$\boldsymbol{\sigma}^k_p = \{\sigma^k_{\alpha\alpha}, \sigma^k_{\beta\beta}, \sigma^k_{\alpha\beta}\}, \qquad \boldsymbol{\sigma}^k_n = \{\sigma^k_{\alpha z}, \sigma^k_{\beta z}, \sigma^k_{zz}\}$$

$$\boldsymbol{\epsilon}^k_p = \{\epsilon^k_{\alpha\alpha}, \epsilon^k_{\beta\beta}, \epsilon^k_{\alpha\beta}\}, \qquad \boldsymbol{\epsilon}^k_n = \{\epsilon^k_{\alpha z}, \epsilon^k_{\beta z}, \epsilon^k_{zz}\}$$

Subscript H denotes stresses evaluated by Hooke's law, whereas subscript G denotes strain from the geometrical relation (7). The subscript d signifies values employed in the displacement formulation. In fact, Eqs. (5) are used in conjunction with a standard displacement formulation, whereas for the adopted mixed-solution procedure, the stress-strain relationships are conveniently put in the following mixed form[18]:

$$\boldsymbol{\sigma}^k_{pH} = \boldsymbol{C}^k_{pp}\boldsymbol{\epsilon}^k_{pG} + \boldsymbol{C}^k_{pn}\boldsymbol{\sigma}^k_{nM}, \qquad \boldsymbol{\epsilon}^k_{nH} = \boldsymbol{C}^k_{np}\boldsymbol{\epsilon}^k_{pG} + \boldsymbol{C}^k_{nn}\boldsymbol{\sigma}^k_{nM} \tag{6}$$

where both stiffness and compliance coefficients are employed. The subscript M states that the transverse stresses are those of the assumed model in Eqs. (8) (see Sec. II.C). The relation between the arrays of coefficients in the two forms of Hooke's law is simply found:

$$\boldsymbol{C}^k_{pp} = \tilde{\boldsymbol{C}}^k_{pp} - \tilde{\boldsymbol{C}}^k_{pn}\tilde{\boldsymbol{C}}^{k^{-1}}_{nn}\tilde{\boldsymbol{C}}^k_{np}, \qquad \boldsymbol{C}^k_{pn} = \tilde{\boldsymbol{C}}^k_{pn}\tilde{\boldsymbol{C}}^{k^{-1}}_{nn}$$

$$\boldsymbol{C}^k_{np} = -\tilde{\boldsymbol{C}}^{k^{-1}}_{nn}\tilde{\boldsymbol{C}}^k_{np}, \qquad \boldsymbol{C}^k_{nn} = \tilde{\boldsymbol{C}}^{k^{-1}}_{nn}$$

Superscript -1 denotes an inversion of the array.

C. Strain Displacements Relations

As one remains within the small deformation field, the strain components $\boldsymbol{\epsilon}^k_p$, $\boldsymbol{\epsilon}^k_n$ are linearly related to the displacements $\boldsymbol{u}^k$ ($\boldsymbol{u}^k = u^k_\alpha, u^k_\beta, u^k_z$), according to the following geometrical relations[51]:

$$\boldsymbol{\epsilon}^k_{pG} = \boldsymbol{D}_p\boldsymbol{u}^k + \boldsymbol{A}_p\boldsymbol{u}^k, \qquad \boldsymbol{\epsilon}^k_{nG} = \boldsymbol{D}_{n\Omega}\boldsymbol{u}^k + \lambda_D\boldsymbol{A}_n\boldsymbol{u}^k + \boldsymbol{D}_{nz}\boldsymbol{u}^k \tag{7}$$

where

$$\boldsymbol{D}_p = \begin{bmatrix} \dfrac{\partial_\alpha}{H^k_\alpha} & 0 & 0 \\ 0 & \dfrac{\partial_\beta}{H^k_\beta} & 0 \\ \dfrac{\partial_\beta}{H^k_\beta} & \dfrac{\partial_\alpha}{H^k_\alpha} & 0 \end{bmatrix}, \qquad \boldsymbol{A}_p = \begin{bmatrix} 0 & 0 & \dfrac{1}{H^k_\alpha R^k_\alpha} \\ 0 & 0 & \dfrac{1}{H^k_\beta R^k_\beta} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{D}_{n\Omega} = \begin{bmatrix} 0 & 0 & \dfrac{\partial_\alpha}{H^k_\alpha} \\ 0 & 0 & \dfrac{\partial_\beta}{H^k_\beta} \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{A}_n = \begin{bmatrix} -\dfrac{1}{H^k_\alpha R^k_\alpha} & 0 & 0 \\ 0 & -\dfrac{1}{H^k_\beta R^k_\beta} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{D}_{nz} = \begin{bmatrix} \partial_z & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & 0 & \partial_z \end{bmatrix}$$

The λ_D is a trace operator. This has been introduced to identify terms that are neglected in the Donnell-type shallow shell theories.[49] No assumptions have been made for the terms H^k_α and H^k_β. The curvature terms have been entirely retained in the following developments.

III. Displacement and Transverse Stress Fields

According to what carried out for plates[2,47] the C^0_z requirements are fulfilled completely and a priori by assuming a layerwise model for both displacement $\boldsymbol{u}^k$ and transverse stress fields $\boldsymbol{\sigma}^k_n$. The following Nth-order expansions have been chosen in the thickness direction of each k layer:

$$\begin{aligned} \boldsymbol{u}^k &= F_t\boldsymbol{u}^k_t + F_b\boldsymbol{u}^k_b + F_r\boldsymbol{u}^k_r = F_\tau\boldsymbol{u}^k_\tau, && \tau = t, b, r; \\ & && r = 2, 3, \ldots, N \\ \boldsymbol{\sigma}^k_{nM} &= F_t\boldsymbol{\sigma}^k_{nt} + F_b\boldsymbol{\sigma}^k_{nb} + F_r\boldsymbol{\sigma}^k_{nr} = F_\tau\boldsymbol{\sigma}^k_{n\tau}, && k = 1, 2, \ldots, N_l \end{aligned} \tag{8}$$

A free parameter of the model is N. Subscripts t and b denote values related to the layer top and bottom surface, respectively. They consist of the linear part of the expansion. Higher-order distributions in the z direction (parabolic, cubic, etc.) are introduced by the r polynomials. The repeated indexes τ are summed over their ranges.

The thickness functions $F_\tau(\zeta_k)$ have been defined by

$$F_t = \frac{P_0 + P_1}{2}, \qquad F_b = \frac{P_0 - P_1}{2}$$

$$F_r = P_r - P_{r-2}, \qquad r = 2, 3, \ldots, N$$

in which $P_j = P_j(\zeta_k)$ is the Legendre polynomial of the jth order ($P_0 = 1$, $P_1 = \zeta_k$, $P_2 = (3\zeta_k^2 - 1)/2, \ldots$) defined in the ζ_k domain: $-1 \le \zeta_k \le 1$. The chosen functions have the following properties:

$$\zeta_k = \begin{cases} 1: F_t = 1, & F_b = 0, \quad F_r = 0 \\ -1: F_t = 0, & F_b = 1, \quad F_r = 0 \end{cases}$$

The top and bottom values have been used as unknown variables. The C^0_z requirements can therefore be easily linked; the compatibility of the displacement and the equilibrium for the transverse stress components read as follows:

$$\begin{aligned} \boldsymbol{u}^k_t &= \boldsymbol{u}^{(k+1)}_b, && k = 1,\ N_l - 1 \\ \boldsymbol{\sigma}^k_{nt} &= \boldsymbol{\sigma}^{(k+1)}_{nb}, && k = 1,\ N_l - 1 \end{aligned} \tag{9}$$

In those cases in which top/bottom-shell stress values are prescribed (zero or imposed values), the following additional C^0_z requirements must be accounted for

$$\boldsymbol{\sigma}^1_{nb} = \bar{\boldsymbol{\sigma}}_{nb}, \qquad \boldsymbol{\sigma}^{N_l}_{nt} = \bar{\boldsymbol{\sigma}}_{nt} \tag{10}$$

where the overbar denotes imposed values in correspondence to the shell boundary. Examples of assumed fields have been plotted in Fig. 2.

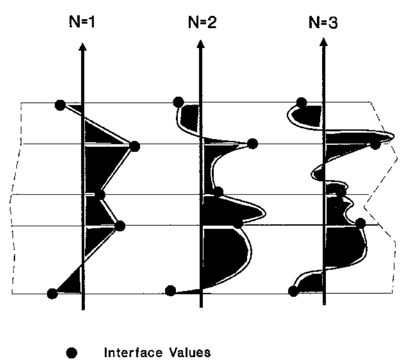

Fig. 2 Assumed displacement and transverse stress fields in the thickness shell direction: linear, parabolic, and cubic cases.

Equations (8) refer to the same order of expansion as the three components for both displacement and transverse stresses. To meet well-known results (see, e.g., Refs. 1, 32, 33, 49, 57 or the author's discussion in Refs. 2 and 44), different polynomial orders should be used in developments that are presented in the subsequent sections. Trace operators that would lead to a shortening of the resulting arrays in Eqs. (18) and (29) could be introduced for this aim. For the sake of brevity, the results related to these aspects have not been discussed. These could be more conveniently reported in future works.

IV. Variational Statements

The displacement approach is formulated in terms of $\boldsymbol{u}^k$ by variationally imposing the equilibrium via the classical principle of virtual displacements. In the static case this states that

$$\sum_{k=1}^{N_l} \int_{\Omega_k} \int_{A_k} \left(\delta \boldsymbol{\epsilon}_{pG}^{k^T} \boldsymbol{\sigma}_{pH_d}^k + \delta \boldsymbol{\epsilon}_{nG}^{k^T} \boldsymbol{\sigma}_{nH_d}^k \right) \mathrm{d}\Omega_k \, \mathrm{d}z = \delta L^e \tag{11}$$

where δ signifies a variational symbol. The variation of the internal work has been split into the in-plane and out-of-plane parts and involves stress from Hooke's law and strain from geometrical relations. The δL_e is the virtual variation of the work carried out by the external surface forces $\boldsymbol{p}^k = \{p_\alpha^k, p_\beta^k, p_z^k\}$.

In the mixed case, equilibrium and compatibility are formulated in terms of the $\boldsymbol{u}^k$ and $\boldsymbol{\sigma}_n^k$ unknowns via Reissner's variational equation[2,39,40,47,58] that in the shell case is herein written as

$$\sum_{k=1}^{N_l} \int_{\Omega_k} \int_{A_k} \left[\delta \boldsymbol{\epsilon}_{pG}^{k^T} \boldsymbol{\sigma}_{pH}^k + \delta \boldsymbol{\epsilon}_{nG}^{k^T} \boldsymbol{\sigma}_{nM}^k + \delta \boldsymbol{\sigma}_{nM}^{k^T} \left(\boldsymbol{\epsilon}_{nG}^k - \boldsymbol{\epsilon}_{nH}^k \right) \right] \mathrm{d}\Omega_k \, \mathrm{d}z$$

$$= \delta L^e \tag{12}$$

The left-hand side includes the variations of the internal work in the shell: The first two terms come from the displacement formulation and lead to variationally consistent equilibrium conditions; the third mixed term variationally enforces the compatibility of the transverse strain components.

As usual in a variational procedure, the derivative of virtual variation of displacement and stress variables moves, by integration of parts, to the corresponding finite variables. To do this, it is convenient to introduce the following array formulas:

$$\int_{\Omega_k} (\boldsymbol{D}_p \delta \boldsymbol{\alpha})^T \boldsymbol{\beta} \, \mathrm{d}\Omega_k = - \int_{\Omega_k} \delta \boldsymbol{\alpha}^T \boldsymbol{D}_p^T \boldsymbol{\beta} \, \mathrm{d}\Omega_k + \int_{\Gamma_k} \boldsymbol{\delta} \boldsymbol{\alpha}^T \boldsymbol{I}_p^T \boldsymbol{\beta} \, \mathrm{d}\Gamma_k$$

$$\int_{\Omega_k} (\boldsymbol{D}_{n\Omega} \delta \boldsymbol{\alpha})^T \boldsymbol{\beta} \, \mathrm{d}\Omega_k = - \int_{\Omega_k} \delta \boldsymbol{\alpha}^T \boldsymbol{D}_\Omega^T \boldsymbol{\beta} \, \mathrm{d}\Omega_k + \int_{\Gamma_k} \boldsymbol{\delta} \boldsymbol{\alpha}^T \boldsymbol{I}_{n\Omega}^T \boldsymbol{\beta} \, \mathrm{d}\Gamma_k \tag{13}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be displacement or stress variables, respectively, and the introduced nondifferentials arrays are

$$\boldsymbol{I}_p = \begin{bmatrix} 1/H_\alpha^k & 0 & 0 \\ 0 & 1/H_\beta^k & 0 \\ 1/H_\beta^k & 1/H_\alpha^k & 0 \end{bmatrix}, \qquad \boldsymbol{I}_{n\Omega} = \begin{bmatrix} 0 & 0 & 1/H_\alpha^k \\ 0 & 0 & 1/H_\beta^k \\ 0 & 0 & 0 \end{bmatrix}$$

The written formulas remain valid if and only if first-order differential operators are involved.

Upon substitution of the introduced displacement expansion, the strains are written in the following form:

$$\boldsymbol{\epsilon}_{pG}^k = F_\tau \boldsymbol{D}_p \boldsymbol{u}_\tau^k + F_\tau \boldsymbol{A}_p \boldsymbol{u}_\tau^k$$

$$\boldsymbol{\epsilon}_{nG}^k = F_\tau \boldsymbol{D}_{n\Omega} \boldsymbol{u}_\tau^k + \lambda_D F_\tau \boldsymbol{A}_n \boldsymbol{u}_\tau^k + F_{\tau_z} \boldsymbol{u}_\tau^k \tag{14}$$

where the further subscript z denotes differentiation.

V. Equilibrium and Constitutive Equations for Mixed Models

In contrast to most of the available shell literature and to the author's previous works related to plates, in the present analysis the definition of stress or strain resultants in the shell thickness direction has been omitted. Such a choice is mainly because of the author's wish to preserve the terms H_α^k, H_β^k in the strain equations (7). In fact, if Love's approximation $H_\alpha^k = H_\beta^k = 1$ is not introduced, as is the case of the present paper, the definition of stress and strain resultants still remains possible, but in the author's opinion, is not convenient. As a result, the governing equations are written directly in terms of the introduced stress and displacement variables in this paper.

Based on the preceding, the Reissner mixed variational equations [Eq. (12)] takes the following form:

$$\sum_{k=1}^{N_l} \Bigg(\int_{\Omega_k} \Big\{ \delta \boldsymbol{u}_\tau^{k^T} \big[\big(-F_\tau \boldsymbol{D}_p^T + F_\tau \boldsymbol{A}_p^T \big) \boldsymbol{C}_{pp} (F_s \boldsymbol{D}_p + F_s \boldsymbol{A}_p) \boldsymbol{u}_s^k$$

$$+ \big(-F_\tau \boldsymbol{D}_p^T + F_\tau \boldsymbol{A}_p^T \big) \boldsymbol{C}_{pn} F_s \boldsymbol{\sigma}_{ns}^k + \big(-F_\tau \boldsymbol{D}_{n\Omega}^T + \lambda_D F_\tau \boldsymbol{A}_n^T$$

$$+ F_{\tau_z} \big) F_s \boldsymbol{\sigma}_{ns}^k \big] \delta \boldsymbol{\sigma}_\tau^{k^T} \big\{ \big[F_\tau F_s \boldsymbol{D}_{n\Omega} + \lambda_D F_\tau F_s \boldsymbol{A}_n + F_\tau F_{s_z}$$

$$- F_\tau \boldsymbol{C}_{np} (F_s \boldsymbol{D}_p + F_s \boldsymbol{A}_p) \big] \boldsymbol{u}_s^k - F_\tau F_s \boldsymbol{C}_{nn} \boldsymbol{\sigma}_{ns}^k \big\} \Big\} \mathrm{d}\Omega_k$$

$$+ \int_{\Gamma_k} \int_{A_k} \delta \boldsymbol{u}_\tau^{k^T} \big[F_\tau \boldsymbol{I}_p^T \boldsymbol{C}_{pp} (F_s \boldsymbol{D}_p + F_s \boldsymbol{A}_p) \boldsymbol{u}_s^k + F_\tau F_s \boldsymbol{I}_p^T \boldsymbol{C}_{pn} \boldsymbol{\sigma}_{ns}^k$$

$$+ F_\tau F_s \boldsymbol{I}_{n\Omega}^T \boldsymbol{\sigma}_{ns}^k \big] \mathrm{d}\Gamma_k \Bigg) = \sum_{k=1}^{N_l} \int_{\Omega_k} \delta \boldsymbol{u}_\tau^{k^T} \boldsymbol{p}_\tau^k \, \mathrm{d}\Omega_k \tag{15}$$

where $\boldsymbol{p}_\tau^k = \{p_{x\tau}^k, p_{y\tau}^k, p_{z\tau}^k\}$ are the variationally consistent load vectors from the applied loadings $\boldsymbol{p}^k$. The case in which both shearing $(p_{\alpha t}^k, p_{\beta t}^k, p_{\alpha b}^k, p_{\beta b}^k)$ and normal (p_{zt}^k, p_{zb}^k) surface forces are applied in correspondence to the top and bottom surface of the layer could be of practical interest:

$$\mathrm{d}\Omega_k^p = \mathrm{d}\Omega_k^t = \big(1 + h_k/2R_\alpha^k\big)\big(1 + h_k/2R_\beta^k\big) \, \mathrm{d}\Omega_k$$

$$\mathrm{d}\Omega_k^p = \mathrm{d}\Omega_k^b = \big(1 - h_k/2R_\alpha^k\big)\big(1 - h_k/2R_\beta^k\big) \, \mathrm{d}\Omega_k$$

By imposing the definition of virtual variations for the unknown stress and displacement variables, the differential system of governing equations and related boundary conditions for the N_l layers in each Ω_k domain are found. The equilibrium and compatibility equations are

$$\delta \boldsymbol{u}_\tau^k \colon \boldsymbol{K}_{uu}^{k\tau s} \boldsymbol{u}_s^k + \boldsymbol{K}_{u\sigma}^{k\tau s} \boldsymbol{\sigma}_{ns}^k = \boldsymbol{p}_\tau^k$$

$$\delta \boldsymbol{\sigma}_{n\tau}^k \colon \boldsymbol{K}_{\sigma u}^{k\tau s} \boldsymbol{u}_s^k + \boldsymbol{K}_{\sigma\sigma}^{k\tau s} \boldsymbol{\sigma}_{ns}^k = 0 \tag{16}$$

with boundary conditions

$$\begin{array}{lcl} \text{geometrical on } \Gamma_k^g & & \text{mechanical on } \Gamma_k^m \\ \boldsymbol{u}_\tau^k = \bar{\boldsymbol{u}}_\tau^k & \text{or} & \boldsymbol{\Pi}_u^{k\tau s} \boldsymbol{u}_s^k + \boldsymbol{\Pi}_\sigma^{k\tau s} \boldsymbol{\sigma}_{ns}^k = \boldsymbol{\Pi}_u^{k\tau s} \bar{\boldsymbol{u}}_s^k + \boldsymbol{\Pi}_\sigma^{k\tau s} \bar{\boldsymbol{\sigma}}_{ns}^k \end{array} \tag{17}$$

in which the overbar denotes assigned values.

The introduced differential arrays are given by the following relations:

$$\boldsymbol{K}_{uu}^{k\tau s} = \int_{A_k} \big(-F_\tau \boldsymbol{D}_p^T + F_\tau \boldsymbol{A}_p^T \big) \boldsymbol{C}_{pp}^k (F_s \boldsymbol{D}_p + F_s \boldsymbol{A}_p) H_\alpha^k H_\beta^k \, \mathrm{d}z$$

$$\boldsymbol{K}_{u\sigma}^{k\tau s} = \int_{A_k} \big[\big(-F_\tau \boldsymbol{D}_p^T + F_\tau \boldsymbol{A}_p^T \big) \boldsymbol{C}_{pn}^k F_s + F_{\tau_z} F_s \boldsymbol{I} + \lambda_D F_\tau F_s \boldsymbol{A}_n$$

$$- F_\tau F_s \boldsymbol{D}_{n\Omega}^T \big] H_\alpha^k H_\beta^k \, \mathrm{d}z$$

$$\boldsymbol{K}_{\sigma u}^{k\tau s} = \int_{A_k} \big[F_\tau F_s \boldsymbol{D}_{n\Omega} + \lambda_D F_\tau F_s \boldsymbol{A}_n + F_\tau F_{s_z} \boldsymbol{I}$$

$$- \boldsymbol{C}_{np}^{k\tau s} (F_\tau F_s \boldsymbol{D}_p + F_\tau F_s \boldsymbol{A}_p) \big] H_\alpha^k H_\beta^k \, \mathrm{d}z$$

$$\boldsymbol{K}_{\sigma\sigma}^{k\tau s} = - \int_{A_k} F_\tau F_s \boldsymbol{C}_{nn}^{k\tau s} H_\alpha^k H_\beta^k \, \mathrm{d}z$$

$$\boldsymbol{\Pi}_u^{k\tau s} = \int_{A_k} F_\tau \boldsymbol{I}_p^T \boldsymbol{C}_{pp} (F_s \boldsymbol{D}_p + F_s \boldsymbol{A}_p) \boldsymbol{I}_p^T H_\alpha^k H_\beta^k \, \mathrm{d}z$$

$$\boldsymbol{\Pi}_\sigma^{k\tau s} = \int_{A_k} \big(F_\tau F_s \boldsymbol{I}_p^T \boldsymbol{C}_{pn} + F_\tau F_s \boldsymbol{I}_{n\Omega}^T \big) H_\alpha^k H_\beta^k \, \mathrm{d}z \tag{18}$$

where $\boldsymbol{I}$ is the unit array. As usual in two-dimensional modelings, the integration in the thickness direction can be carried out a priori by introducing the following layer integrals:

$$\left(J^{k\tau s}, J_\alpha^{k\tau s}, J_\beta^{k\tau s}, J_{\alpha/\beta}^{k\tau s}, J_{\beta/\alpha}^{k\tau s}, J_{\alpha\beta}^{k\tau s}\right) = \int_{A_k} F_\tau F_s \left(1, H_\alpha^k, H_\beta^k, \frac{H_\alpha^k}{H_\beta^k}, \frac{H_\beta^k}{H_\alpha^k}, H_\alpha^k H_\beta^k\right) \mathrm{d}z$$

$$\left(J^{k\tau_z s}, J_\alpha^{k\tau_z s}, J_\beta^{k\tau_z s}, J_{\alpha\beta}^{k\tau_z s}\right) = \int_{A_k} F_{\tau_z} F_s \left(1, H_\alpha^k, H_\beta^k, H_\alpha^k H_\beta^k\right) \mathrm{d}z$$

$$\left(J^{k\tau s_z}, J_\alpha^{k\tau s_z}, J_\beta^{k\tau s_z}, J_{\alpha\beta}^{k\tau s_z}\right) = \int_{A_k} F_\tau F_{s_z} \left(1, H_\alpha^k, H_\beta^k, H_\alpha^k H_\beta^k\right) \mathrm{d}z$$

$$\left(J^{k\tau_z s_z}, J_{\alpha\beta}^{k\tau_z s_z}\right) = \int_{A_k} F_{\tau_z} F_{s_z} \left(1, H_\alpha^k H_\beta^k\right) \mathrm{d}z \tag{19}$$

The integrals involved in the displacement formulations (presented in Sec. VI) have also been introduced.

The explicit forms of the differential operators in Eqs. (18) follow. First, the pure stiffness part $\boldsymbol{K}_{uu}^{k\tau s}$ is expanded:

$$K_{uu_{11}}^{k\tau s} = -C_{pp_{33}}^k J_{\alpha/\beta}^{k\tau s} \partial_{\beta\beta} - 2C_{pp_{13}}^k J^{k\tau s} \partial_{\alpha\beta} - C_{pp_{11}}^k J_{\beta/\alpha}^{k\tau s} \partial_{\alpha\alpha}$$

$$K_{uu_{12}}^{k\tau s} = -C_{pp_{32}}^k J_{\alpha/\beta}^{k\tau s} \partial_{\beta\beta} - C_{pp_{12}}^k J^{k\tau s} \partial_{\alpha\beta} - C_{pp_{33}}^k J^{k\tau s} \partial_{\beta\alpha} - C_{pp_{13}}^k J_{\beta/\alpha}^{k\tau s} \partial_{\alpha\alpha}$$

$$K_{uu_{13}}^{k\tau s} = -\frac{1}{R_\beta^k}\left(C_{pp_{32}}^k J_{\alpha/\beta}^{k\tau s} \partial_\beta + C_{pp_{12}}^k J^{k\tau s} \partial_\alpha\right) - \frac{1}{R_\alpha^k}\left(C_{pp_{31}}^k J^{k\tau s} \partial_\beta + C_{pp_{11}}^k J_{\beta/\alpha}^{k\tau s} \partial_\alpha\right)$$

$$K_{uu_{21}}^{k\tau s} = K_{uu_{12}}^{k\tau s}$$

$$K_{uu_{22}}^{k\tau s} = -C_{pp_{33}}^k J_{\beta/\alpha}^{k\tau s} \partial_{\alpha\alpha} - 2C_{pp_{23}}^k J^{k\tau s} \partial_{\beta\alpha} - C_{pp_{22}}^k J_{\alpha/\beta}^{k\tau s} \partial_{\beta\beta}$$

$$K_{uu_{23}}^{k\tau s} = -\frac{1}{R_\alpha^k}\left(C_{pp_{31}}^k J_{\beta/\alpha}^{k\tau s} \partial_\alpha + C_{pp_{21}}^k J^{k\tau s} \partial_\beta\right) - \frac{1}{R_\beta^k}\left(C_{pp_{32}}^k J^{k\tau s} \partial_\alpha + C_{pp_{22}}^k J_{\alpha/\beta}^{k\tau s} \partial_\beta\right)$$

$$K_{uu_{31}}^{k\tau s} = -K_{uu_{13}}^{k\tau s}, \qquad K_{uu_{32}}^{k\tau s} = -K_{uu_{23}}^{k\tau s}$$

$$K_{uu_{33}}^{k\tau s} = \frac{1}{R_\beta^{k^2}} C_{pp_{22}}^k J_{\alpha/\beta}^{k\tau s} + \frac{2}{R_\alpha^k R_\beta^k} C_{pp_{12}}^k J^{k\tau s} + \frac{1}{R_\alpha^{k^2}} C_{pp_{11}}^k J_{\beta/\alpha}^{k\tau s} \tag{20}$$

The mixed stiffness-compliance arrays $\boldsymbol{K}_{us}^{k\tau s}$ leads to

$$K_{us_{11}}^{k\tau s} = J_{\alpha\beta}^{k\tau_z s} - \left(\lambda_D / R_\alpha^k\right) J_\beta^{k\tau s}, \qquad K_{us_{12}}^{k\tau s} = 0$$

$$K_{us_{13}}^{k\tau s} = -C_{pn_{33}}^k J_\alpha^{k\tau s} \partial_\beta - C_{pn_{13}}^k J_\beta^{k\tau s} \partial_\alpha, \qquad K_{us_{21}}^{k\tau s} = 0$$

$$K_{us_{22}}^{k\tau s} = J_{\alpha\beta}^{k\tau_z s} - \left(\lambda_D / R_\beta^k\right) J_\alpha^{k\tau s}$$

$$K_{us_{23}}^{k\tau s} = -C_{pn_{33}}^k J_\beta^{k\tau s} \partial_\alpha - C_{pn_{23}}^k J_\alpha^{k\tau s} \partial_\beta$$

$$K_{us_{31}}^{k\tau s} = -J_\beta^{k\tau s} \partial_\alpha, \qquad K_{us_{32}}^{k\tau s} = -J_\alpha^{k\tau s} \partial_\beta$$

$$K_{us_{33}}^{k\tau s} = J_{\alpha\beta}^{k\tau_z s} + \left(1/R_\alpha^k\right) C_{pn_{13}}^k J_\beta^{k\tau s} + \left(1/R_\beta^k\right) C_{pn_{23}}^k J_\alpha^{k\tau s} \tag{21}$$

The mixed compliance-stiffness array $\boldsymbol{K}_{su}^{k\tau s}$ leads to

$$K_{su_{11}}^{k\tau s} = J_{\alpha\beta}^{k\tau s_z} - \left(\lambda_D / R_\alpha^k\right) J_\beta^{k\tau s}, \qquad K_{su_{12}}^{k\tau s} = 0$$

$$K_{su_{13}}^{k\tau s} = J_\beta^{k\tau s} \partial_\alpha, \qquad K_{su_{21}}^{k\tau s} = 0$$

$$K_{su_{22}}^{k\tau s} = J_{\alpha\beta}^{k\tau s_z} - \left(\lambda_D / R_\beta^k\right) J_\alpha^{k\tau s}, \qquad K_{su_{23}}^{k\tau s} = J_\alpha^{k\tau s} \partial_\beta$$

$$K_{su_{31}}^{k\tau s} = -C_{np_{33}}^k J_\alpha^{k\tau s} \partial_\beta - C_{np_{31}}^k J_\beta^{k\tau s} \partial_\alpha$$

$$K_{su_{32}}^{k\tau s} = -C_{np_{32}}^k J_\alpha^{k\tau s} \partial_\beta - C_{np_{33}}^k J_\beta^{k\tau s} \partial_\alpha$$

$$K_{su_{33}}^{k\tau s} = J_{\alpha\beta}^{k\tau s_z} - \left(1/R_\alpha^k\right) C_{pn_{31}}^k J_\beta^{k\tau s} - \left(1/R_\beta^k\right) C_{pn_{32}}^k J_\alpha^{k\tau s} \tag{22}$$

The pure compliance array $\boldsymbol{K}_{ss}^{k\tau s}$ is

$$K_{ss_{ij}}^{k\tau s} = -C_{nn_{ij}}^k J_{\alpha\beta}^{k\tau s_z}, \qquad i, j = 1, 2, 3 \tag{23}$$

The boundary array $\boldsymbol{\Pi}_u^{k\tau s}$ leads to

$$\Pi_{u_{11}}^{k\tau s} = C_{pp_{11}}^k J_{\beta/\alpha}^{k\tau s} \partial_\alpha + C_{pp_{13}}^k J^{k\tau s} (\partial_\alpha + \partial_\beta) + C_{pp_{33}}^k J_{\alpha/\beta}^{k\tau s} \partial_\beta$$

$$\Pi_{u_{12}}^{k\tau s} = C_{pp_{13}}^k J_{\beta/\alpha}^{k\tau s} \partial_\alpha + C_{pp_{33}}^k J^{k\tau s} \partial_\alpha + C_{pp_{12}}^k J^{k\tau s} \partial_\beta + C_{pp_{32}}^k J_{\alpha/\beta}^{k\tau s} \partial_\beta$$

$$\Pi_{u_{13}}^{k\tau s} = \left(1/R_\alpha^k\right)\left(C_{pp_{11}}^k J_{\beta/\alpha}^{k\tau s} + C_{pp_{31}}^k J^{k\tau s}\right) + \left(1/R_\beta^k\right)\left(C_{pp_{32}}^k J_{\alpha/\beta}^{k\tau s} + C_{pp_{12}}^k J^{k\tau s}\right)$$

$$\Pi_{u_{21}}^{k\tau s} = C_{pp_{31}}^k J_{\beta/\alpha}^{k\tau s} \partial_\alpha + C_{pp_{21}}^k J^{k\tau s} \partial_\alpha + C_{pp_{33}}^k J^{k\tau s} \partial_\beta + C_{pp_{23}}^k J_{\alpha/\beta}^{k\tau s} \partial_\beta$$

$$\Pi_{u_{22}}^{k\tau s} = C_{pp_{33}}^k J_{\beta/\alpha}^{k\tau s} \partial_\alpha + C_{pp_{23}}^k J^{k\tau s} (\partial_\alpha + \partial_\beta) + C_{pp_{22}}^k J_{\alpha/\beta}^{k\tau s} \partial_\beta$$

$$\Pi_{u_{23}}^{k\tau s} = \left(1/R_\alpha^k\right)\left(C_{pp_{31}}^k J_{\beta/\alpha}^{k\tau s} + C_{pp_{12}}^k J^{k\tau s}\right) + \left(1/R_\beta^k\right)\left(C_{pp_{22}}^k J_{\alpha/\beta}^{k\tau s} + C_{pp_{32}}^k J^{k\tau s}\right)$$

$$\Pi_{u_{31}}^{k\tau s} = \Pi_{u_{32}}^{k\tau s} = \Pi_{u_{33}}^{k\tau s} = 0 \tag{24}$$

Finally, the expansion of the boundary array $\boldsymbol{\Pi}_s^{k\tau s}$ leads to

$$\Pi_{\sigma_{11}}^{k\tau s} = \Pi_{\sigma_{12}}^{k\tau s} = 0, \qquad \Pi_{\sigma_{13}}^{k\tau s} = C_{pn_{13}}^k J_\beta^{k\tau s} + C_{pn_{33}}^k J_\alpha^{k\tau s}$$

$$\Pi_{\sigma_{21}}^{k\tau s} = \Pi_{\sigma_{22}}^{k\tau s} = 0, \qquad \Pi_{\sigma_{23}}^{k\tau s} = C_{pn_{33}}^k J_\beta^{k\tau s} + C_{pn_{32}}^k J_\alpha^{k\tau s}$$

$$\Pi_{\sigma_{31}}^{k\tau s} = J_\beta^{k\tau s} \partial_\alpha, \qquad \Pi_{\sigma_{32}}^{k\tau s} = J_\alpha^{k\tau s}\ \partial_\beta, \qquad \Pi_{\sigma_{33}}^{k\tau s} = 0 \tag{25}$$

Cylindrical shell equations are simply obtained by enforcing $R_\alpha = \infty$ (or $R_\beta = \infty$) whereas spherical shell geometries correspond to the case $R_\alpha = R_\beta$. The corresponding Donnell's shallow-shell equations are obtained by discarding the terms multiplied by the trace operators λ_D. Neglecting all of the curvature terms, the governing equations, written for multilayered plates[46,47] are obtained as particular cases.

VI. Equilibrium Equations for Displacement Formulations

Following the same procedure developed for the mixed case, Eq. (11) leads to

$$\sum_{k=1}^{N_l} \Bigg(\int_{\Omega_k} \int_{A_k} \delta \boldsymbol{u}_\tau^{k^T} \Big\{ \left(-F_\tau \boldsymbol{D}_p^T + F_\tau \boldsymbol{A}_p^T\right) \Big[\tilde{\boldsymbol{C}}_{pp} (F_s \boldsymbol{D}_p + F_s \boldsymbol{A}_p) + \tilde{\boldsymbol{C}}_{pp} \left(F_s \boldsymbol{D}_{n\Omega} + \lambda_D F_s \boldsymbol{A}_n + F_{s_z}\right) \Big] + \left(-F_\tau \boldsymbol{D}_{n\Omega}^T + \lambda_D F_\tau \boldsymbol{A}_n^T + F_{\tau_z}\right) \Big[\tilde{\boldsymbol{C}}_{np} (F_s \boldsymbol{D}_p + F_s \boldsymbol{A}_p) + \tilde{\boldsymbol{C}}_{nn} \left(F_s \boldsymbol{D}_{n\Omega} + \lambda_D F_s \boldsymbol{A}_n + F_{s_z}\right) \Big] \Big\} \boldsymbol{u}_s \,\mathrm{d}\Omega_k + \int_{\Gamma_k} \int_{A_k} \delta \boldsymbol{u}_\tau^{k^T} \Big\{ F_\tau \boldsymbol{I}_p^T \Big[\tilde{\boldsymbol{C}}_{pp} (F_s \boldsymbol{D}_p + F_s \boldsymbol{A}_p) + \tilde{\boldsymbol{C}}_{pp} \left(F_s \boldsymbol{D}_{n\Omega} + \lambda_D F_s \boldsymbol{A}_n + F_{s_z}\right) \Big] + F_\tau \boldsymbol{I}_{n\Omega}^T \Big[\tilde{\boldsymbol{C}}_{np} (F_s \boldsymbol{D}_p + F_s \boldsymbol{A}_p) + \tilde{\boldsymbol{C}}_{nn} \left(F_s \boldsymbol{D}_{n\Omega} + \lambda_D F_s \boldsymbol{A}_n + F_{s_z}\right) \Big] \Big\} \boldsymbol{u}_s \,\mathrm{d}\Gamma_k \Bigg) = \sum_{k=1}^{N_l} \int_{\Omega_k} \delta \boldsymbol{u}_\tau^{k^T} \boldsymbol{p}_\tau^k \,\mathrm{d}\Omega_k^p \tag{26}$$

By imposing the definition of virtual variations for the unknown displacements, the differential system of governing equations and related boundary conditions follow. For the N_l layers the equilibrium equations on each Ω^k domain are

$$\delta \boldsymbol{u}_\tau^k:\ \boldsymbol{K}_d^{k\tau s}\boldsymbol{u}_s^k = \boldsymbol{p}_\tau^k \tag{27}$$

The boundary conditions on Γ^k are

$$\begin{array}{lcl} \text{geometrical on } \Gamma_k^g & & \text{mechanical on } \Gamma_k^m \\ \boldsymbol{u}_\tau^k = \bar{\boldsymbol{u}}_\tau^k & \text{or} & \boldsymbol{\Pi}_d^{k\tau s}\boldsymbol{u}_s^k = \boldsymbol{\Pi}_d^{k\tau s}\bar{\boldsymbol{u}}_s^k \end{array} \tag{28}$$

The introduced differential arrays are

$$\begin{aligned}
\boldsymbol{K}_d^{k\tau s} &= \int_{A_k} \big\{\big(-F_\tau \boldsymbol{D}_p^T + F_\tau \boldsymbol{A}_p^T\big)\big[\tilde{\boldsymbol{C}}_{pp}(F_s\boldsymbol{D}_p + F_s\boldsymbol{A}_p) \\
&\quad + \tilde{\boldsymbol{C}}_{pn}\big(F_s\boldsymbol{D}_{n\Omega} + \lambda_D F_s\boldsymbol{A}_n + F_{s_z}\big)\big] \\
&\quad + \big(-F_\tau \boldsymbol{D}_{n\Omega}^T + \lambda_D F_\tau \boldsymbol{A}_n^T + F_{\tau_z}\big)\big[\tilde{\boldsymbol{C}}_{np}(F_s\boldsymbol{D}_p + F_s\boldsymbol{A}_p) \\
&\quad + \tilde{\boldsymbol{C}}_{nn}\big(F_s\boldsymbol{D}_{n\Omega} + F_s\boldsymbol{A}_n + F_{s_z}\big)\big]\big\} H_\alpha^k H_\beta^k \,\mathrm{d}z_k \\
\boldsymbol{\Pi}_d^{k\tau s} &= \int_{A_k} \big\{F_\tau \boldsymbol{I}_p^T\big[\tilde{\boldsymbol{C}}_{pp}(F_s\boldsymbol{D}_p + F_s\boldsymbol{A}_p) \\
&\quad + \tilde{\boldsymbol{C}}_{pp}\big(F_s\boldsymbol{D}_{n\Omega} + \lambda_D F_s\boldsymbol{A}_n + F_{s_z}\big)\big] + F_\tau \boldsymbol{I}_{n\Omega}^T\big[\tilde{\boldsymbol{C}}_{np}(F_s\boldsymbol{D}_p + F_s\boldsymbol{A}_p) \\
&\quad + \tilde{\boldsymbol{C}}_{nn}\big(F_s\boldsymbol{D}_{n\Omega} + \lambda_D F_s\boldsymbol{A}_n + F_{s_z}\big)\big]\big\} H_\alpha^k H_\beta^k \,\mathrm{d}z_k
\end{aligned} \tag{29}$$

By using the integrals defined in Eqs. (19), the nine elements of the differential operators in the 3×3 arrays of Eqs. (29) can be explicitly written. First the stiffness array $\boldsymbol{K}_d^{k\tau s}$ is written

$$\begin{aligned}
K_{d_{11}}^{k\tau s} &= \tilde{C}_{nn_{11}}^k J_{\alpha\beta}^{k\tau_z s_z} + \lambda_D \frac{1}{R_\alpha^k}\tilde{C}_{nn_{11}}^k\left(-J_\beta^{k\tau_z s} - J_\beta^{k\tau s_z} + \frac{1}{R_\alpha^k}J_{\beta\alpha}^{k\tau s}\right) \\
&\quad - \tilde{C}_{pp_{11}}^k J_{\beta/\alpha}^{k\tau s}\partial_{\alpha\alpha} - 2\tilde{C}_{pp_{13}}^k J^{k\tau s}\partial_{\alpha\beta} - \tilde{C}_{pp_{33}}^k J_{\alpha/\beta}^{k\tau s}\partial_{\beta\beta} \\
K_{d_{12}}^{k\tau s} &= \tilde{C}_{nn_{12}}^k J_{\alpha\beta}^{k\tau_z s_z} + \lambda_D \tilde{C}_{nn_{12}}^k\left(-\frac{1}{R_\beta^k}J_\alpha^{k\tau_z s} - \frac{1}{R_\alpha^k}J_\beta^{k\tau s_z} \right. \\
&\quad \left. + \frac{1}{R_\alpha^k R_\beta^k}J^{k\tau s}\right) - \tilde{C}_{pp_{12}}^k J^{k\tau s}\partial_{\alpha\beta} - \tilde{C}_{pp_{13}}^k J_{\beta/\alpha}^{k\tau s}\partial_{\alpha\alpha} \\
&\quad - \tilde{C}_{pp_{32}}^k J_{\alpha/\beta}^{k\tau s}\partial_{\beta\beta} + \tilde{C}_{pp_{33}}^k J_{\alpha\beta}^{k\tau s} \\
K_{d_{13}}^{k\tau s} &= \tilde{C}_{nn_{11}}^k\left(J_\beta^{k\tau_z s}\partial_\alpha - \lambda_D \frac{1}{R_\alpha^k}J_{\beta/\alpha}^{k\tau s}\partial_\alpha\right) \\
&\quad + \tilde{C}_{nn_{12}}^k\left(J_\alpha^{k\tau_z s}\partial_\beta - \lambda_D \frac{1}{R_\alpha^k}J^{k\tau s}\partial_\beta\right) - \tilde{C}_{pn_{13}}^k J_\beta^{k\tau s_z}\partial_\alpha \\
&\quad - \tilde{C}_{pn_{33}}^k J_\alpha^{k\tau s_z}\partial_\beta - \frac{1}{R_\alpha^k}\big(\tilde{C}_{pp_{11}}^k J_{\beta/\alpha}^{k\tau s}\partial_\alpha + \tilde{C}_{pp_{31}}^k J^{k\tau s}\partial_\beta\big) \\
&\quad - \frac{1}{R_\beta^k}\big(\tilde{C}_{pp_{12}}^k J^{k\tau s}\partial_\alpha + \tilde{C}_{pp_{32}}^k J_{\alpha/\beta}^{k\tau s}\partial_\beta\big) \\
K_{d_{21}}^{k\tau s} &= K_{d_{12}}^{k\tau s} \\
K_{d_{22}}^{k\tau s} &= \tilde{C}_{nn_{22}}^k J_{\alpha\beta}^{k\tau_z s_z} + \lambda_D \frac{1}{R_\beta^k}\tilde{C}_{nn_{22}}^k\left(-J_\alpha^{k\tau_z s} - J_\alpha^{k\tau s_z} + \frac{1}{R_\beta^k}J_{\alpha\beta}^{k\tau s}\right) \\
&\quad - \tilde{C}_{pp_{22}}^k J_{\alpha/\beta}^{k\tau s}\partial_{\beta\beta} - \tilde{C}_{pp_{23}}^k J^{k\tau s}(\partial_{\beta\alpha} + \partial_{\alpha\beta}) - \tilde{C}_{pp_{33}}^k J_{\beta/\alpha}^{k\tau s}\partial_{\beta\beta} \\
K_{d_{23}}^{k\tau s} &= \tilde{C}_{nn_{22}}^k\left(J_\alpha^{k\tau_z s}\partial_\beta - \lambda_D \frac{1}{R_\beta^k}J^{k\tau s}\partial_\beta\right) \\
&\quad + \tilde{C}_{nn_{21}}^k\left(J_\beta^{k\tau_z s}\partial_\alpha - \lambda_D \frac{1}{R_\beta^k}J_{\alpha/\beta}^{k\tau s}\partial_\alpha\right) - \tilde{C}_{pn_{23}}^k J_\alpha^{k\tau s_z}\partial_\beta \\
&\quad - \tilde{C}_{pn_{33}}^k J_\beta^{k\tau s_z}\partial_\alpha - \frac{1}{R_\beta^k}\tilde{C}_{pp_{22}}^k J_{\alpha/\beta}^{k\tau s}\partial_\beta - \frac{1}{R_\beta^k}\tilde{C}_{pp_{32}}^k J^{k\tau s}\partial_\alpha \\
&\quad - \frac{1}{R_\alpha^k}\tilde{C}_{pp_{21}}^k J^{k\tau s}\partial_\beta - \frac{1}{R_\alpha^k}\tilde{C}_{pp_{31}}^k J_{\beta/\alpha}^{k\tau s}\partial_\alpha \\
K_{d_{31}}^{k\tau s} &= -K_{d_{13}}^{k\tau s}, \qquad K_{d_{32}}^{k\tau s} = -K_{d_{23}}^{k\tau s} \\
K_{d_{33}}^{k\tau s} &= -\tilde{C}_{nn_{11}}^k J_{\beta/\alpha}^{k\tau s}\partial_{\alpha\alpha} - 2\tilde{C}_{nn_{12}}^k J^{k\tau s}\partial_{\alpha\beta} - \tilde{C}_{nn_{22}}^k J_{\alpha/\beta}^{k\tau s}\partial_{\beta\beta} \\
&\quad + \tilde{C}_{nn_{33}}^k J_{\alpha\beta}^{k\tau_z s_z} + \frac{1}{R_\alpha^k}\left(\frac{1}{R_\alpha^k}\tilde{C}_{pp_{11}}^k J_{\beta/\alpha}^{k\tau s} + \tilde{C}_{pp_{31}}^k J_\beta^{k\tau_z s} + \tilde{C}_{pp_{13}}^k J_\beta^{k\tau s_z}\right) \\
&\quad + \frac{2}{R_\alpha^k R_\beta^k}J_\alpha^{k\tau s}\tilde{C}_{pp_{12}}^k + \frac{1}{R_\beta^k}\left(\frac{1}{R_\beta^k}\tilde{C}_{pp_{22}}^k J_{\alpha/\beta}^{k\tau s} + \tilde{C}_{pp_{32}}^k J_\alpha^{k\tau_z s} \right. \\
&\quad \left. + \tilde{C}_{pp_{23}}^k J_\alpha^{k\tau s_z}\right)
\end{aligned} \tag{30}$$

The boundary array $\boldsymbol{\Pi}_d^{k\tau s}$ leads to

$$\begin{aligned}
\Pi_{d_{11}}^{k\tau s} &= \tilde{C}_{pp_{11}}^k J_{\beta/\alpha}^{k\tau s}\partial_\alpha + \tilde{C}_{pp_{13}}^k J^{k\tau s}(\partial_\alpha + \partial_\beta) + \tilde{C}_{pp_{33}}^k J_{\alpha/\beta}^{k\tau s}\partial_\beta \\
\Pi_{d_{12}}^{k\tau s} &= \tilde{C}_{pp_{13}}^k J_{\beta/\alpha}^{k\tau s}\partial_\alpha + \tilde{C}_{pp_{33}}^k J^{k\tau s}\partial_\alpha + \tilde{C}_{pp_{12}}^k J^{k\tau s}\partial_\beta + \tilde{C}_{pp_{32}}^k J_{\alpha/\beta}^{k\tau s}\partial_\beta \\
\Pi_{d_{13}}^{k\tau s} &= \tilde{C}_{pp_{13}}^k J_\beta^{k\tau s_z} + \tilde{C}_{pp_{33}}^k J_\alpha^{k\tau s_z} + \big(1/R_\alpha^k\big)\big(\tilde{C}_{pp_{11}}^k J_{\beta/\alpha}^{k\tau s} \\
&\quad + \tilde{C}_{pp_{31}}^k J^{k\tau s}\big) + \big(1/R_\beta^k\big)\big(\tilde{C}_{pp_{32}}^k J_{\alpha/\beta}^{k\tau s} + \tilde{C}_{pp_{12}}^k J^{k\tau s}\big) \\
\Pi_{d_{21}}^{k\tau s} &= \tilde{C}_{pp_{31}}^k J_{\beta/\alpha}^{k\tau s}\partial_\alpha + \tilde{C}_{pp_{21}}^k J^{k\tau s}\partial_\alpha + \tilde{C}_{pp_{33}}^k J^{k\tau s}\partial_\beta + \tilde{C}_{pp_{23}}^k J_{\alpha/\beta}^{k\tau s}\partial_\beta \\
\Pi_{d_{22}}^{k\tau s} &= \tilde{C}_{pp_{33}}^k J_{\beta/\alpha}^{k\tau s}\partial_\alpha + \tilde{C}_{pp_{23}}^k J^{k\tau s}(\partial_\alpha + \partial_\beta) + \tilde{C}_{pp_{22}}^k J_{\alpha/\beta}^{k\tau s}\partial_\beta \\
\Pi_{d_{23}}^{k\tau s} &= \tilde{C}_{pp_{33}}^k J_\beta^{k\tau s_z} + \tilde{C}_{pp_{23}}^k J_\alpha^{k\tau s_z} + \big(1/R_\alpha^k\big)\big(\tilde{C}_{pp_{31}}^k J_{\beta/\alpha}^{k\tau s} \\
&\quad + \tilde{C}_{pp_{21}}^k J^{k\tau s}\big) + \big(1/R_\beta^k\big)\big(\tilde{C}_{pp_{22}}^k J_{\alpha/\beta}^{k\tau s} + \tilde{C}_{pp_{32}}^k J^{k\tau s}\big) \\
\Pi_{d_{31}}^{k\tau s} &= \tilde{C}_{nn_{11}}^k J_\beta^{k\tau s_z} - \big(\lambda_D/R_\alpha^k\big)\big(\tilde{C}_{nn_{11}}^k J_{\beta/\alpha}^{k\tau s} + \tilde{C}_{nn_{21}}^k J^{k\tau s}\big) + \tilde{C}_{nn_{21}}^k J_\alpha^{k\tau s_z} \\
\Pi_{d_{32}}^{k\tau s} &= \tilde{C}_{nn_{12}}^k J_\beta^{k\tau s_z} - \big(\lambda_D/R_\beta^k\big)\big(\tilde{C}_{nn_{22}}^k J_{\alpha/\beta}^{k\tau s} + \tilde{C}_{nn_{12}}^k J^{k\tau s}\big) + \tilde{C}_{nn_{22}}^k J_\alpha^{k\tau s_z} \\
\Pi_{d_{33}}^{k\tau s} &= \tilde{C}_{pp_{11}}^k J_{\beta/\alpha}^{k\tau s}\partial_\alpha + \tilde{C}_{nn_{12}}^k J^{k\tau s}(\partial_\alpha + \partial_\beta) + \tilde{C}_{nn_{22}}^k J_{\alpha/\beta}^{k\tau s}\partial_\beta
\end{aligned} \tag{31}$$

VII. Fulfillment of C_z^0 Requirements and Governing Equations at a Multilayered Level

In the preceding sections the governing shell equations have been considered to be independent of the N_l layers. Interlaminar continuity conditions, i.e., C_z^0 requirements in Eqs. (9) and (10) must be used to drive the governing equations for the whole multilayered shell. This can be done as follows.

1) The governing equations are written for each k layer. In contrast to plates, it is essential to have the same product of type $\mathrm{d}\alpha^*\,\mathrm{d}\beta^*$ in each k-layer equation for shell geometry because of the curvature changes. This can be done by choosing the value on the reference shell surface Ω $\mathrm{d}\alpha\,\mathrm{d}\beta$ as $\mathrm{d}\alpha^*\,\mathrm{d}\beta^*$. Each $\mathrm{d}\alpha_k\,\mathrm{d}\beta_k$ defined on Ω_k is therefore written as

$$\mathrm{d}\alpha_k\,\mathrm{d}\beta_k = \frac{R_\alpha R_\beta}{R_\alpha^k R_\beta^k}\,\mathrm{d}\alpha\,\mathrm{d}\beta$$

2) Each set of $3N \times 3N$ arrays defined in one of Eqs. (18) and (29) is then put in a kth-order array in such a way that those parts related to the top values of the variables are on the top of that array. For instance, one of the kth-layer arrays of Eqs. (18) or (29) is considered. Such an array is shown in Fig. 3. The nine contributions to this array have been denoted in Fig. 3 by two subscripts related to the top, the r variables, and bottom terms. These terms are listed in the following order: t, r, b.

3) The k-layer arrays are set in a larger, multilayer array by ordering the layers from the plate top $k = N_l$ to the plate bottom $k = 1$.

4) The interface continuity conditions in Eqs. (9) are imposed by moving the layer arrays along the diagonal of the multilayer array. As a result one obtains the array drawn in the lower part of Fig. 3.

Table 1 Connectivity arrays: parabolic expansion in the kth layer[a]

k	Top	Resultant	Bottom
1	3	2	1
2	5	4	3
3	7	6	5
..	..	..	..
$k-1$	$2k-1$	$2k-2$	$2k-3$
k	$2k+1$	$2k$	$2k-1$
$k+1$	$2k+3$	$2k+2$	$2k+1$
..	..	..	..
N_l	$2N_l+1$	$2N_l+1$	$2N_l-1$

[a] $N = 2$.

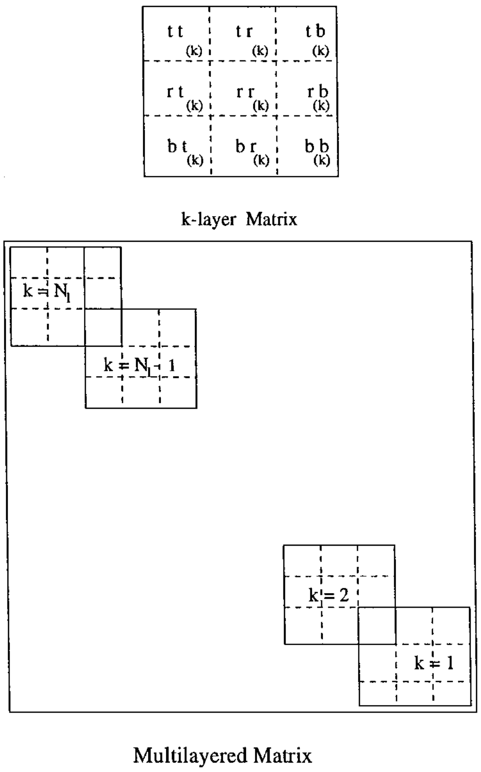

Fig. 3 Assembly of matrices from layers to a multilayered level.

5) The remaining C_z^0 requirements in Eqs. (10), wherever they occur, can be directly imposed to the multilayer arrays. The dimension of the same arrays and the number of unknown variables reduces on multiplication of these arrays for the known top-bottom shell-stress values. In most general cases in which a stress or displacement is imposed, such a multiplication produces loading arrays.

This procedure is common to each discretization method in which interface variables are used as unknowns. For instance, this procedure can be seen as a classical assembly technique of finite element methods.[59] In this case each layer is considered as one element: interface and internal nodes are introduced in correspondence to top-bottom and r variables, respectively (see Fig. 4). A standard connectivity array can then be introduced; an example is given in Table 1 where the case of parabolic expansion has been considered. Such a technique remains very useful for the implementation. It has in fact been coded for the numerical application presented in Ref. 55.

The following shell arrays for the displacement unknowns is introduced at the very end:

$$\boldsymbol{u} = \Big\{\boldsymbol{u}_b^{1^T}, \boldsymbol{u}_t^{1^T}, \boldsymbol{u}_r^{1^T}; \boldsymbol{u}_t^{2^T}, \boldsymbol{u}_r^{2^T}; \ldots \boldsymbol{u}_t^{k^T}, \boldsymbol{u}_r^{k^T};$$

$$\boldsymbol{u}_t^{(k+1)^T}, \boldsymbol{u}_r^{(k+1)^T}; \ldots \boldsymbol{u}_t^{(N_l-1)^T}, \boldsymbol{u}_r^{(N_l-1)^T}; \boldsymbol{u}_t^{N_l^T}, \boldsymbol{u}_r^{N_l^T}\Big\} \tag{32}$$

In which the top and r variables have been chosen as layer unknowns. The governing equations and boundary conditions of the multilayered shells, related to the displacement formulation, is for-

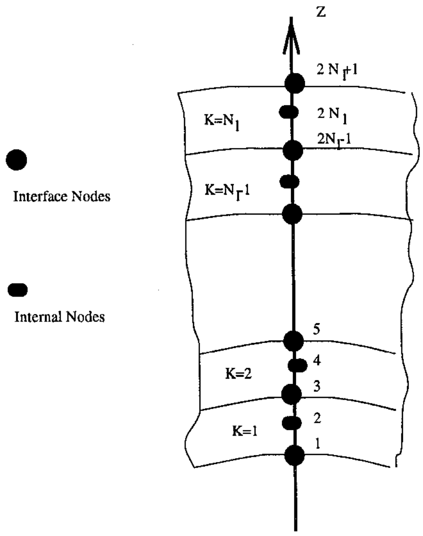

Fig. 4 Multilayered shell: interface and internal nodes for the case $N = 2$.

mally written in the following form:

$$\boldsymbol{K}_d \boldsymbol{u} = \boldsymbol{p}, \qquad \boldsymbol{u} = \bar{\boldsymbol{u}} \quad \text{or} \quad \boldsymbol{\Pi}_d \boldsymbol{u} = \boldsymbol{\Pi}_d \bar{\boldsymbol{u}} \tag{33}$$

The following further array of the multilayered transverse stress unknowns is introduced for the mixed case:

$$\boldsymbol{\sigma}_n = \Big\{\boldsymbol{\sigma}_{nt}^{1^T}, \boldsymbol{\sigma}_{nr}^{1^T}; \boldsymbol{\sigma}_{nt}^{2^T}, \boldsymbol{\sigma}_{nr}^{2^T}; \ldots \boldsymbol{\sigma}_{nt}^{k^T}, \boldsymbol{\sigma}_{nr}^{k^T};$$

$$\boldsymbol{\sigma}_{nt}^{(k+1)^T}, \boldsymbol{\sigma}_{nr}^{(k+1)^T}; \ldots \boldsymbol{\sigma}_{nt}^{(N_l-1)^T}, \boldsymbol{\sigma}_{nr}^{(N_l-1)^T}; \boldsymbol{\sigma}_{nr}^{N_l^T}\Big\} \tag{34}$$

Therefore, the governing system of differential equations at a multilayered level related to the mixed formulations is formally written in the following final form:

$$\boldsymbol{K}_{uu}\boldsymbol{u} + \boldsymbol{K}_{u\sigma}\boldsymbol{\sigma}_n = \boldsymbol{p} + \boldsymbol{p}_u^{1N_l}, \qquad \boldsymbol{K}_{\sigma u}\boldsymbol{u} + \boldsymbol{K}_{\sigma\sigma}\boldsymbol{\sigma}_n = \boldsymbol{p}_\sigma^{1N_l} \tag{35}$$

while the boundary conditions are

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \quad \text{or} \quad \boldsymbol{\Pi}_u\boldsymbol{u} + \boldsymbol{\Pi}_\sigma\boldsymbol{\sigma}_n = \boldsymbol{\Pi}_u\bar{\boldsymbol{u}} + \boldsymbol{\Pi}_\sigma\bar{\boldsymbol{\sigma}}_n + \boldsymbol{q}_\sigma^{1N_l} \tag{36}$$

where $\boldsymbol{p}_u^{1N_l}$, $\boldsymbol{p}_\sigma^{1N_l}$, and $\boldsymbol{q}_\sigma^{1N_l}$ are arrays coming from transverse stress values imposed at the top/bottom of the shell equation (10).

VIII. Conclusion

By employing a Reissner's mixed variational equation, this paper has derived the differential equations that govern the equilibrium and compatibility of layered, double-curved shells modeled using mixed layerwise theories. Classical displacement formulations and related equivalent single-layer equations have also been derived for comparison purposes. The following points can be remarked:

1) With respect to existing theories, the proposed models permit one to fulfill completely and a priori the continuity conditions at the interfaces for both displacement and transverse stress components (transverse displacement and transverse normal stress included); i.e., no postprocessing technique is required to compute transverse stresses.

2) The order of the expansion N in the thickness direction has been taken as a free parameter. The use of top and bottom values as unknown layer variables permits the use of the continuity conditions to write easily the governing equations from a layer to a multilayered level.

3) No assumptions have been made concerning the terms related to the thickness-to-radii h/R shell ratio. The equations related to Donnell's shallow shell-type approximations have been derived for all of the considered theories.

The assessment and results are presented in the companion paper,[55] where the excellent agreement to the exact three-dimensional solution, which had previously been found for laminated plates, is confirmed.

Appendix A: ESLM Equations for Displacement Formulation

Sections V–VII have presented the governing equations for the mixed and classical displacement formulation. These equations have been written at layer level. As in classical shell analysis, the assumed model Eqs. (8) can also be written at a multilayered level:

$$\boldsymbol{u} = F_t\boldsymbol{u}_t + F_b\boldsymbol{u}_b + F_r\boldsymbol{u}_r = F_\tau\boldsymbol{u}_\tau, \qquad \tau = t,\, b,\, r; \quad r = 2, 3, \ldots, N$$

$$\text{¾}_{nM} = F_t\text{¾}_{nt} + F_b\text{¾}_{nb} + F_r\text{¾}_{nr} = F_\tau\text{¾}_{n\tau} \tag{A.1}$$

where the functions F_τ are expressed in terms of the shell coordinate $\zeta = 2z/h$ and correspondent ESLM theories can be derived. It appears difficult to extend the ESLM description to the mixed cases whenever the homogeneous conditions on the transverse stress have to be imposed at the shell bottom and top surfaces. A weak form of Hooke's law could be more conveniently used in mixed ESLM analysis.[2] This section has been restricted to displacement formulation and could serve, as in Ref. 55, to compare classical ESLM results to layerwise analysis. Furthermore, the use of top and bottom values as variables is not mandatory for standard displacement formulation. Classical Taylor expansions are more convenient for this purpose. In such a case $F_t = 1$, $F_b = z$, $F_r = z^r$; $\boldsymbol{u}^t$ becomes the displacement vector of a shell point lying on Ω and $\boldsymbol{u}^b$, $\boldsymbol{u}^r$, their higher-order derivatives.

By repeating what was done for the layerwise cases of Sec. VI, the governing equations are formally written in the same manner. In fact, it is sufficient to introduce the following multilayered 3×3 stiffness arrays:

$$\left(\boldsymbol{E}^{\tau s}_{ij}, \boldsymbol{E}^{\tau s}_{\alpha_{ij}}, \boldsymbol{E}^{\tau s}_{\beta_{ij}}, \boldsymbol{E}^{\tau s}_{(\alpha/\beta)_{ij}}, \boldsymbol{E}^{\tau s}_{(\beta/\alpha)_{ij}}, \boldsymbol{E}^{\tau s}_{\alpha\beta_{ij}}\right) = \sum_{k=1}^{N_l} \tilde{\boldsymbol{C}}^k_{ij}\left(J^{k\tau s}, J^{k\tau s}_\alpha, J^{k\tau s}_\beta, J^{k\tau s}_{\alpha/\beta}, J^{k\tau s}_{\beta/\alpha}, J^{k\tau s}_{\alpha\beta}\right)$$

$$\left(\boldsymbol{E}^{\tau_z s}_{ij}, \boldsymbol{E}^{\tau_z s}_{\alpha_{ij}}, \boldsymbol{E}^{\tau_z s}_{\beta_{ij}}, \boldsymbol{E}^{\tau_z s}_{\alpha\beta_{ij}}\right) = \sum_{k=1}^{N_l} \tilde{\boldsymbol{C}}^k_{ij}\left(J^{k\tau_z s}, J^{k\tau_z s}_\alpha, J^{k\tau_z s}_\beta, J^{k\tau_z s}_{\alpha\beta}\right)$$

$$\left(\boldsymbol{E}^{\tau s_z}_{ij}, \boldsymbol{E}^{\tau s_z}_{\alpha_{ij}}, \boldsymbol{E}^{\tau s_z}_{\beta_{ij}}, \boldsymbol{E}^{\tau s_z}_{\alpha\beta_{ij}}\right) = \sum_{k=1}^{N_l} \tilde{\boldsymbol{C}}^k_{ij}\left(J^{k\tau s_z}, J^{k\tau s_z}_\alpha, J^{k\tau s_z}_\beta, J^{k\tau s_z}_{\alpha\beta}\right)$$

$$\left(\boldsymbol{E}^{k\tau_z s_z}_{ij}, \boldsymbol{E}^{k\tau_z s_z}_{\alpha\beta_{ij}}\right) = \sum_{k=1}^{N_l} \tilde{\boldsymbol{C}}^k_{ij}\left(J^{k\tau_z s_z}, J^{k\tau_z s_z}_{\alpha\beta}\right) \tag{A.2}$$

They consist of a 3×3 array for each τ, s couple and they result in a four-index array. The index coupled ij can assume the four expressions pp, pn, np, nn and are related to Hooke's law in Eqs. (6). It is intended that both function F_τ, F_s and terms H^k_α and H^k_β in J^k-type integrals have to be considered as functions of the whole shell coordinate z for ESLM cases.

The differential operators for the displacement formulated theories assume the following explicit expressions:

$$K^{\tau s}_{d_{11}} = E^{\tau_z s_z}_{\alpha\beta_{nn_{11}}} + \frac{\lambda_D}{R_\alpha}\left(-E^{\tau_z s}_{\beta_{nn_{11}}} - E^{\tau s_z}_{\beta_{nn_{11}}} + \frac{1}{R_\alpha}E^{\tau s}_{(\beta\alpha)_{nn_{11}}}\right) - E^{\tau s}_{(\beta/\alpha)_{pp_{11}}}\partial_{\alpha\alpha} - 2E^{\tau s}_{pp_{13}}\partial_{\alpha\beta} - E^{\tau s}_{(\alpha/\beta)_{pp_{33}}}\partial_{\beta\beta}$$

$$K^{\tau s}_{d_{12}} = E^{\tau_z s_z}_{\alpha\beta_{nn_{12}}} + \lambda_D\left(-\frac{1}{R_\beta}E^{\tau_z s}_{\alpha_{nn_{12}}} - \frac{1}{R_\alpha}E^{\tau s_z}_{\beta_{nn_{12}}} + \frac{1}{R_\alpha R_\beta}E^{\tau s}_{nn_{12}}\right) - E^{\tau s}_{pp_{12}}\partial_{\alpha\beta} - E^{\tau s}_{(\beta/\alpha)_{pp_{13}}}\partial_{\alpha\alpha} - E^{\tau s}_{(\alpha/\beta)_{pp_{32}}}\partial_{\beta\beta} + E^{\tau s}_{\alpha\beta_{pp_{33}}}$$

$$K^{\tau s}_{d_{13}} = \left(E^{\tau_z s}_{\beta_{nn_{11}}}\partial_\alpha - \lambda_D\frac{1}{R_\alpha}E^{\tau s}_{(\beta/\alpha)_{nn_{11}}}\partial_\alpha\right) + \left(E^{\tau_z s}_{\alpha_{nn_{12}}}\partial_\beta - \lambda_D\frac{1}{R_\alpha}E^{\tau s}_{nn_{12}}\partial_\beta\right) - E^{\tau s_z}_{\beta_{pn_{13}}}\partial_\alpha - E^{\tau s_z}_{\alpha_{pn_{33}}}\partial_\beta - \frac{1}{R_\alpha}\left(E^{\tau s}_{(\beta/\alpha)_{pp_{11}}}\partial_\alpha + E^{\tau s}_{pp_{31}}\partial_\beta\right) - \frac{1}{R_\beta}\left(E^{\tau s}_{pp_{12}}\partial_\alpha + E^{\tau s}_{(\alpha/\beta)_{pp_{32}}}\partial_\beta\right)$$

$$K^{\tau s}_{d_{21}} = K^{\tau s}_{d_{12}}$$

$$K^{\tau s}_{d_{22}} = E^{\tau_z s_z}_{\alpha\beta_{nn_{22}}} + \lambda_D\frac{1}{R_\beta}\left(-E^{\tau_z s}_{\alpha_{nn_{22}}} - E^{\tau s_z}_{\alpha_{nn_{22}}} + \frac{1}{R_\beta}E^{\tau s}_{\alpha\beta_{nn_{22}}}\right) - E^{\tau s}_{(\alpha/\beta)_{pp_{22}}}\partial_{\beta\beta} - E^{\tau s}_{pp_{23}}(\partial_{\beta\alpha} + \partial_{\alpha\beta}) - E^{\tau s}_{(\beta/\alpha)_{pp_{33}}}\partial_{\beta\beta}$$

$$K^{\tau s}_{d_{23}} = \left(E^{\tau_z s}_{\alpha_{nn_{22}}}\partial_\beta - \lambda_D\frac{1}{R_\beta}E^{\tau s}_{nn_{22}}\partial_\beta\right) + \left(E^{\tau_z s}_{\beta_{nn_{21}}}\partial_\alpha - \lambda_D\frac{1}{R_\beta}E^{\tau s}_{(\alpha/\beta)_{nn_{21}}}\partial_\alpha\right) - E^{\tau s_z}_{\alpha_{pn_{23}}}\partial_\beta - E^{\tau s_z}_{\beta_{pn_{23}}}\partial_\alpha - \frac{1}{R_\beta}E^{\tau s}_{(\alpha/\beta)_{pp_{22}}}\partial_\beta - \frac{1}{R_\beta}E^{\tau s}_{pp_{32}}\partial_\alpha - \frac{1}{R_\alpha}E^{\tau s}_{pp_{21}}\partial_\beta - \frac{1}{R_\alpha}\tilde{E}^{\tau s}_{(\beta/\alpha)_{pp_{22}}}\partial_\alpha$$

$$K^{\tau s}_{d_{31}} = -K^{\tau s}_{d_{13}}, \qquad K^{\tau s}_{d_{32}} = -K^{\tau s}_{d_{23}}$$

$$K^{\tau s}_{d_{33}} = -E^{\tau s}_{(\beta/\alpha)_{nn_{11}}}\partial_{\alpha\alpha} - 2E^{\tau s}_{nn_{12}}\partial_{\alpha\beta} - E^{\tau s}_{(\alpha/\beta)_{nn_{22}}}\partial_{\beta\beta} + E^{\tau_z s_z}_{\alpha\beta_{nn_{33}}} + \frac{1}{R_\alpha}\left(\frac{1}{R_\alpha}E^{\tau s}_{(\beta/\alpha)_{pp_{11}}} + E^{\tau_z s}_{\beta_{pp_{31}}} + E^{\tau s_z}_{\beta_{pp_{13}}}\right) + \frac{2}{R^k_\alpha R_\beta}E^{\tau s}_{\alpha_{pp_{12}}} + \frac{1}{R_\beta}\left(\frac{1}{R_\beta}E^{\tau s}_{(\alpha/\beta)_{pp_{22}}} + E^{\tau_z s}_{\alpha_{pp_{32}}} + E^{\tau s_z}_{\alpha_{pp_{23}}}\right) \tag{A.3}$$

Boundary terms could be written in the same manner but are not quoted for the sake of brevity. The numerical analysis development in Ref. 55 requires us to express the work carried out by external pressure applied at the top $\boldsymbol{p}_t$ and at the bottom $\boldsymbol{p}_b$ shell surface:

$$\delta L_e = \int_\Omega \boldsymbol{p}_t\delta\bar{\boldsymbol{u}}_t\,\mathrm{d}\Omega_t + \int_\Omega \boldsymbol{p}_b\delta\bar{\boldsymbol{u}}_b\,\mathrm{d}\Omega_b \tag{A.4}$$

where

$$\mathrm{d}\Omega_t = (1 + h/2R_\alpha)[1 + (h/2R_\beta)\,\mathrm{d}\Omega]$$

$$\mathrm{d}\Omega_b = (1 - h/2R_\alpha)[1 - (h/2R_\beta)\,\mathrm{d}\Omega]$$

$$\bar{\boldsymbol{u}}_t = \boldsymbol{u}_t + (h/2)^r\boldsymbol{u}_r, \qquad \bar{\boldsymbol{u}}_b = \boldsymbol{u}_b + (-h/2)^r\boldsymbol{u}_r$$

It follows that

$$\delta L_e = \int_\Omega\left\{\boldsymbol{p}_t\left[\delta\boldsymbol{u}_t + \left(\frac{h}{2}\right)^r\delta\boldsymbol{u}_r\right]\left(1 + \frac{h}{2R_\alpha}\right)\left(1 + \frac{h}{2R_\beta}\right) + \boldsymbol{p}_b\left[\delta\boldsymbol{u}_b + \left(-\frac{h}{2}\right)^r\delta\boldsymbol{u}_r\right]\left(1 - \frac{h}{2R_\alpha}\right)\left(1 - \frac{h}{2R_\beta}\right)\right\}\mathrm{d}\Omega \tag{A.5}$$

Appendix B: Example Showing How Indicial Notation Works

To show how indicial notation works, the case in which a Taylor linear expansion is used for the ESLM equations derived in Appendix A ($F_t = 1$, $F_b = z$) can be considered. The 21 different terms of the 6×6 symmetric stiffness array $\boldsymbol{K}^{\tau s}_d$ ($\tau, s = t, b$) are given next. For the sake of simplicity the case $H_\alpha = H_\beta = 1$ (Love's approximations) is considered. In such a case

$$(F_tF_t,\ F_tF_b,\ F_bF_b) = (1,\ z,\ z^2)$$

It follows that the stiffnesses in Eqs. (A.2) are simply replaced by the well-known membrane, coupling, and bending[60] stiffnesses:

$$\left(A_{pp_{ij}}, B_{pp_{ij}}, D_{pp_{ij}}\right) = \int_{-h/2}^{h/2}\left(F_tF_t, F_tF_b, F_bF_b\right)Qpp_{ij}\,\mathrm{d}z$$

$$\left(A_{pn_{ij}}, B_{pn_{ij}}, D_{pn_{ij}}\right) = \int_{-h/2}^{h/2}\left(F_tF_t, F_tF_b, F_bF_b\right)Qpn_{ij}\,\mathrm{d}z$$

$$\left(A_{nn_{ij}}, B_{nn_{ij}}, D_{nn_{ij}}\right) = \int_{-h/2}^{h/2}\left(F_tF_t, F_tF_b, F_bF_b\right)Qnn_{ij}\,\mathrm{d}z$$

Therefore, Eqs. (A.3) lead to

$$K^{tt}_{d_{11}} = \frac{\lambda_D}{R^2_\alpha} A_{nn_{11}} - A_{pp_{11}}\partial_{\alpha\alpha} - 2A_{pp_{13}}\partial_{\alpha\beta} - A_{pp_{33}}\partial_{\beta\beta}$$

$$K^{tt}_{d_{12}} = \frac{\lambda_D}{R_\alpha R_\beta} A_{nn_{12}} - A_{pp_{12}}\partial_{\alpha\beta} - A_{pp_{13}}\partial_{\alpha\alpha} - A_{pp_{32}}\partial_{\beta\beta} + A_{pp_{33}}$$

$$K^{tt}_{d_{13}} = -\frac{\lambda_D}{R_\alpha} A_{nn_{11}}\partial_\alpha - \lambda_D\frac{1}{R_\alpha} A_{nn_{12}}\partial_\beta - \frac{1}{R_\alpha}\left(A_{pp_{11}}\partial_\alpha + A_{pp_{31}}\partial_\beta\right)$$
$$- \frac{1}{R_\beta}\left(A_{pp_{12}}\partial_\alpha + A_{pp_{32}}\partial_\beta\right)$$

$$K^{tt}_{d_{22}} = \frac{\lambda_D}{R^2_\beta} A_{nn_{22}} - A_{pp_{22}}\partial_{\beta\beta} - A_{pp_{23}}(\partial_{\beta\alpha} + \partial_{\alpha\beta}) - A_{pp_{33}}\partial_{\beta\beta}$$

$$K^{tt}_{d_{23}} = -\frac{\lambda_D}{R_\beta} A_{nn_{22}}\partial_\beta - \frac{\lambda_D}{R_\beta} A_{nn_{21}}\partial_\alpha - \frac{1}{R_\beta} A_{pp_{22}}\partial_\beta - \frac{1}{R_\beta} A_{pp_{32}}\partial_\alpha$$
$$- \frac{1}{R_\alpha} A_{pp_{21}}\partial_\beta - \frac{1}{R_\alpha} A_{pp_{22}}\partial_\alpha$$

$$K^{tt}_{d_{33}} = -A_{nn_{11}}\partial_{\alpha\alpha} - 2A_{nn_{12}}\partial_{\alpha\beta} - A_{nn_{22}}\partial_{\beta\beta} + \frac{1}{R^2_\alpha} A_{pp_{11}}$$
$$+ \frac{2}{R_\alpha R_\beta} A_{pp_{12}} + \frac{1}{R^2_\beta} A_{pp_{22}}$$

$$K^{tb}_{d_{11}} = \frac{\lambda_D}{R_\alpha}\left(-A_{nn_{11}} + \frac{1}{R_\alpha} E_{nn_{11}}\right) - B_{pp_{11}}\partial_{\alpha\alpha}$$
$$-2B_{pp_{13}}\partial_{\alpha\beta} - B_{pp_{33}}\partial_{\beta\beta}$$

$$K^{tb}_{d_{12}} = \lambda_D\left(\frac{1}{R_\alpha R_\beta} A_{nn_{12}} + \frac{1}{R_\alpha R_\beta} B_{nn_{12}}\right) - B_{pp_{12}}\partial_{\alpha\beta} - B_{pp_{13}}\partial_{\alpha\alpha}$$
$$- B_{pp_{32}}\partial_{\beta\beta} + B_{pp_{33}}$$

$$K^{tb}_{d_{13}} = -\lambda_D\frac{1}{R_\alpha} B_{nn_{11}}\partial_\alpha - \lambda_D\frac{1}{R_\alpha} B_{nn_{12}}\partial_\beta - A_{pn_{13}}\partial_\alpha - A_{pn_{33}}\partial_\beta$$
$$- \frac{1}{R_\alpha}\left(B_{pp_{11}}\partial_\alpha + B_{pp_{31}}\partial_\beta\right) - \frac{1}{R_\beta}\left(B_{pp_{12}}\partial_\alpha + B_{pp_{32}}\partial_\beta\right)$$

$$K^{\tau s}_{d_{21}} = K^{\tau s}_{d_{12}}$$

$$K^{tb}_{d_{22}} = \lambda_D\frac{1}{R_\beta}\left(-A_{nn_{22}} + \frac{1}{R_\beta} B_{nn_{22}}\right) - B_{pp_{22}}\partial_{\beta\beta}$$
$$- B_{pp_{23}}(\partial_{\beta\alpha} + \partial_{\alpha\beta}) - B_{pp_{33}}\partial_{\beta\beta}$$

$$K^{tb}_{d_{23}} = -\lambda_D\frac{1}{R_\beta} B_{nn_{22}}\partial_\beta - \lambda_D\frac{1}{R_\beta} B_{nn_{21}}\partial_\alpha - A_{pn_{23}}\partial_\beta - A_{pn_{23}}\partial_\alpha$$
$$- \frac{1}{R_\beta} B_{pp_{22}}\partial_\beta - \frac{1}{R_\beta} B_{pp_{32}}\partial_\alpha - \frac{1}{R_\alpha} B_{pp_{21}}\partial_\beta - \frac{1}{R_\alpha}\tilde{B}_{pp_{22}}\partial_\alpha$$

$$K^{tb}_{d_{31}} = -K^{tb}_{d_{13}}, \qquad K^{tb}_{d_{32}} = -K^{tb}_{d_{23}}$$

$$K^{tb}_{d_{33}} = -B_{nn_{11}}\partial_{\alpha\alpha} - 2B_{nn_{12}}\partial_{\alpha\beta} - B_{nn_{22}}\partial_{\beta\beta} + \frac{1}{R_\alpha}\left(\frac{1}{R_\alpha} B_{pp_{11}}\right.$$
$$\left. + A_{pp_{13}}\right) + \frac{2}{R_\alpha R_\beta} B_{pp_{12}} + \frac{1}{R_\beta}\left(\frac{1}{R_\beta} B_{pp_{22}} + A_{pp_{23}}\right)$$

$$K^{bb}_{d_{11}} = A_{nn_{11}} + \frac{\lambda_D}{R_\alpha}\left(-B_{nn_{11}} - B_{nn_{11}} + \frac{1}{R_\alpha} D_{nn_{11}}\right) - D_{pp_{11}}\partial_{\alpha\alpha}$$
$$- 2D_{pp_{13}}\partial_{\alpha\beta} - D_{pp_{33}}\partial_{\beta\beta}$$

$$K^{bb}_{d_{12}} = A_{nn_{12}} + \lambda_D\left(-\frac{1}{R_\beta} B_{nn_{12}} - \frac{1}{R_\alpha} B_{nn_{12}} + \frac{1}{R_\alpha R_\beta} D_{nn_{12}}\right)$$
$$- D_{pp_{12}}\partial_{\alpha\beta} - D_{pp_{13}}\partial_{\alpha\alpha} - D_{pp_{32}}\partial_{\beta\beta} + D_{pp_{33}}$$

$$K^{bb}_{d_{13}} = B_{nn_{11}}\partial_\alpha - \lambda_D\frac{1}{R_\alpha} D_{nn_{11}}\partial_\alpha + B_{nn_{12}}\partial_\beta - \lambda_D\frac{1}{R_\alpha} D_{nn_{12}}\partial_\beta$$
$$- B_{pn_{13}}\partial_\alpha - B_{pn_{33}}\partial_\beta - \frac{1}{R_\alpha}\left(D_{pp_{11}}\partial_\alpha + D_{pp_{31}}\partial_\beta\right)$$
$$- \frac{1}{R_\beta}\left(D_{pp_{12}}\partial_\alpha + D_{pp_{32}}\partial_\beta\right)$$

$$K^{bb}_{d_{22}} = A_{nn_{22}} + \lambda_D\frac{1}{R_\beta}\left(-B_{nn_{22}} - B_{nn_{22}} + \frac{1}{R_\beta} D_{nn_{22}}\right) - D_{pp_{22}}\partial_{\beta\beta}$$
$$- D_{pp_{23}}(\partial_{\beta\alpha} + \partial_{\alpha\beta}) - D_{pp_{33}}\partial_{\beta\beta}$$

$$K^{bb}_{d_{23}} = \left(B_{nn_{22}}\partial_\beta - \lambda_D\frac{1}{R_\beta} D_{nn_{22}}\partial_\beta\right) + \left(B_{nn_{21}}\partial_\alpha - \lambda_D\frac{1}{R_\beta} D_{nn_{21}}\partial_\alpha\right)$$
$$- B_{pn_{23}}\partial_\beta - B_{pn_{23}}\partial_\alpha - \frac{1}{R_\beta} D_{pp_{22}}\partial_\beta - \frac{1}{R_\beta} D_{pp_{32}}\partial_\alpha$$
$$- \frac{1}{R_\alpha} D_{pp_{21}}\partial_\beta - \frac{1}{R_\alpha}\tilde{D}_{pp_{22}}\partial_\alpha$$

$$K^{bb}_{d_{33}} = -D_{nn_{11}}\partial_{\alpha\alpha} - 2D_{nn_{12}}\partial_{\alpha\beta} - D_{nn_{22}}\partial_{\beta\beta} + A_{nn_{33}}$$
$$+ \frac{1}{R_\alpha}\left(\frac{1}{R_\alpha} D_{pp_{11}} + B_{pp_{31}} + B_{pp_{13}}\right) + \frac{2}{R^k_\alpha R_\beta} D_{pp_{12}}$$
$$+ \frac{1}{R_\beta}\left(\frac{1}{R_\beta} D_{pp_{22}} + B_{pp_{32}} + B_{pp_{23}}\right) \tag{B.1}$$

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